Quiz 8; Wednesday, October 18
MATH 110 with Professor Stankova
Section 114; 5-6 pm
GSI: Saad Qadeer

Solutions

You have 10 minutes to complete the quiz. Calculators are not permitted. Please include all relevant calculations and explanations (unless stated otherwise).

1. (3 + 9 points) Let V and W be finite-dimensional spaces and let $T : V \to W$ and $S : W \to V$ be linear.

   (a) Show that $(TS - I_W)$ is not invertible if and only if 1 is an eigenvalue of TS.

   As $(TS - I_w) : W \to W$, it fails to be invertible if and only if it has a non-trivial kernel, i.e., some non-zero $w \in W$ such that

   $$(TS - I_W)(w) = 0 \Leftrightarrow (TS)(w) = w.$$

   Thus, $(TS - I_W)$ is not invertible if and only if 1 is an eigenvalue of TS.

   (b) Prove that if $(TS - I_W)$ fails to be invertible, then so does $(ST - I_V)$.

   Suppose that $(TS - I_W)$ is not invertible. From (a), it follows that TS has an eigenvalue 1, with corresponding eigenvector w: $(TS)(w) = w$. Note that Sw cannot be zero, otherwise w would be zero as well. Applying S on both sides gives $(ST)(Sw) = Sw$ so Sw is an eigenvector of ST with eigenvalue 1. Using (a) again shows that $(ST - I_V)$ must fail to be invertible.

2. (1 + 1 + 1 points) Mark each statement as True or False. You do not need to show your work but a blank answer is worth 0 points and an incorrect answer is worth -1 point.

   (a) If the row-reduced echelon form of a matrix A is the identity matrix, then A is diagonalizable.

   False: Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

   (b) Let A be an $n \times n$ matrix. If

   $$(A - 2I)(A - 3I)(A - 4I) = 0,$$

   then at least one of $2, 3, 4$ is an eigenvalue of A.

   True: Take determinants throughout; at lease one of $\det(A - 2I)$, $\det(A - 3I)$, $\det(A - 4I)$ must be zero.

   (c) A lower triangular matrix with distinct diagonal entries is always diagonalizable.

   True: Distinct diagonal entries ensure that the eigenvalues are distinct as well so the matrix is diagonalizable.